Does the continuum theory of dynamic fracture work?

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We investigate the validity of the linear elastic fracture mechanics approach to dynamic fracture. We first test the predictions in a lattice simulation, using a formula of Eshelby for the time-dependent stress intensity factor. Excellent agreement with the theory is found. We then use the same method to analyze the experiment of Sharon and Fineberg. The data here are not consistent with the theoretical expectation.

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There has been renewed interest in the physics community concerning the problem of dynamic fracture. This interest was kindled by experiments [1,2] showing a universal transition to crack branching and the realization that one could not approach this question within the confines of traditional fracture mechanics [3]. Since then, there have been other novel experimental findings [4] as well as further evidence of theoretical inadequacies.

In the standard approach (see, for example, Ref. [5]), dubbed linear elastic fracture mechanics (LEFM), one assumes that continuum elasticity is valid everywhere outside of an microscopically sized process zone. The instantaneous crack tip velocity is then postulated to depend only on the singular part of the stress field obtained by solving this macroscopic continuum problem. This singularity of the stress field is universal in nature, up to an overall multiplicative factor, the so-called stress-intensity factor (SIF). Even if there is a well-defined relationship between the crack velocity and the SIF, the theory cannot predict the form of the relationship, as that depends explicitly on physics at the scale of the process zone. As realized initially by Slepyan [6], one way to remedy this deficiency is to model the entire system as a lattice of mass points connected by nonlinear springs. On scales larger compared to the lattice spacing, the displacement approaches that predicted by the continuum theory; on the scale of the crack tip, the stress field divergence is regularized, thereby allowing for the imposition of a physically sensible breaking criterion. This criterion is usually in the form of a critical spring displacement, as this allows for the possibility of analytical solutions of the model [7–12].

Thus, the lattice model provides a self-consistent realization of the basic assumption underlying engineering fracture mechanics, the ability to separate the linear elasticity calculation from the microscopic physics controlling the tip. In general, however, it is hard to test the LEFM since it is difficult to reliably measure the SIF in a lattice calculation, as this requires an extremely (and impractically) large system. Only then is there an appreciable range of scales for which one is both sufficiently far from the crack tip that lattice effects are unimportant and sufficiently close that the fairly weak square-root singularity dominates. However, for the case of a crack accelerating from rest, the approach of Kostrov [13] and Eshelby [14] provides an analytic prediction of the SIF, independent of the details of the dynamics. With this first-principles determination of the SIF, it is possible to test the LEFM picture. Specifically, we will show that in the lattice model, in accord with expectations, the stress-intensity factor governing the strength of the stress singularity is the only information passed from the macroscopic field to the process-zone dynamics of the crack tip. In fact, this separation is quantitatively accurate even for rather small lattices where one might have questioned the efficacy of the continuum approach. Moreover, these results allow us to construct a test of the theory based on the actual fracture data presented by Sharon and Fineberg [15] for the same case of a crack accelerating from rest. Here, however, the data do not appear to conform to theoretical expectations. At the end, we discuss possible avenues for resolving the conflict.

We begin with the lattice model. We work with a two-dimensional (2D) square lattice with unit spacing between the mass points. These masses are coupled via both nearest-neighbor (nn) and next-nearest neighbor (nnn) ideally brittle central force springs with spring constants k_1 and k_2, respectively. It is easy to show that with the choice $k_1=2k_2=2\mu=2\lambda_{2d}$, the continuum linear elastic limit of this model is isotropic with the aforementioned Lame' constants; hence the Poisson ratio $\nu_{2d}=1/4$. We will assume that the actual 3D system exhibits plane stress and so can be approximated via a 2D system with $\lambda_{2D}=2\mu\lambda/(2\mu+\lambda)$; the actual Poisson ratio of the material being modeled is $\nu=\nu_{2D}/(1-\nu_{2D})=1/3$. The dynamics arising from this force is taken to include the possibility of a Kelvin viscosity term. The final equation of motion is therefore

$$\frac{\partial^2\vec{u}(\vec{x})}{\partial t^2}=\left(1+\eta\frac{\partial}{\partial t}\right)\left[\frac{k_1}{2}\sum_{\hat{n}\in nn}\{[\vec{u}(\vec{x}+\hat{n})-\vec{u}(\vec{x})]\cdot\hat{n}\}\hat{n}+\frac{k_2}{2}\sum_{\hat{n}\in nnn}\{[\vec{u}(\vec{x}+\hat{n})-\vec{u}(\vec{x})]\cdot\hat{n}\}\hat{n}\right]. \quad (1)$$

As discussed in Refs. [8,16], the damping due to nonzero Kelvin viscosity occurs only inside the process zone if η is

chosen 0(1). Finally, any bond whose length goes above a breaking threshold, which we take to be 1, has its spring constant set to zero.

As discussed above, we study in detail a finite length crack accelerating from rest. Initially, a crack is placed along the midline of a sample (here between rows 0 and 1 of our lattice), extending a length ℓ_0 from the left edge. The top and bottom rows of the lattice have fixed (and opposite) displacements $\vec{u}=\Delta\hat{y}$ and the lateral edges are free. The loading is chosen to be just below the critical value at which the crack will start to propagate; since this loading is a decreasing function of crack length, a crack with one additional broken bond will in fact start to elongate. The system is then allowed to fully relax to its equilibrium stress state; this is accomplished via a multigrid technique described in detail elsewhere [17]. Once this is done, an additional bond is broken by hand (i.e., its spring constant is set to zero; no actual displacement of particles is involved) and the crack tip accelerates. As it moves, we monitor the bonds across the crack surface $y=1/2$. When all three bonds attached to a given point $\vec{x}=(x,1)$ break, the crack length is deemed to have increased by 1 and the velocity at that time is measured as the inverse of the time interval since the last such event. We do not allow bonds off the crack line to break, thereby suppressing any possible branching instabilities. We also enforce symmetry across the crack surface, simulating only the upper half of the lattice. Each run is characterized by the transverse lattice size W, the initial length of the crack ℓ_0, damping constant η, and the driving displacement Δ. Typical data generated by this procedure for both the undamped and highly damped cases are presented in Fig. 1.

According to the classic calculation by Eshelby [14], the stress intensity factor K or equivalently the Eshelby B factor (which is a constant multiple of K), at the progressing crack tip for a system that starts in equilibrium can, up to a certain time, be written as a product of two factors:

$$B_I(t)=A(v(t))\int_{\ell_0}^{\ell(t)} dx\,\frac{\sigma_{eq}^{yy}(x)}{\sqrt{x-\ell_0}}. \tag{2}$$

Here σ_{eq}^{yy} is the normal stress on the midline $y=1/2$ as found from the equilibrium stress field; it diverges of course near the edge of the equilibrium crack $x=\ell_0$ with a static-stress intensity factor. The first factor $A(v)$ depends only on the *instantaneous* velocity at time t, and the second factor (which we will refer to as B_0) depends on time only through the *instantaneous* crack length $\ell(t)$. This equation holds true as long as sound waves reflected from the boundaries have not interacted with the crack tip. There are two boundaries one must be concerned with, the lateral boundaries at $y=\pm W/2$ and the edge at $x=0$. We take care that all our data come from times before these interactions occur.

Now, if the true microscopic breaking events only couple to the macroscopic field via B_I, we expect that there will be some fixed relationship between the tip velocity and this number. Usually, this relationship is considered to arise due to energy conservation, being written as an equality of the energy flux into the tip and the energy necessary to break bonds along a unit length of crack,

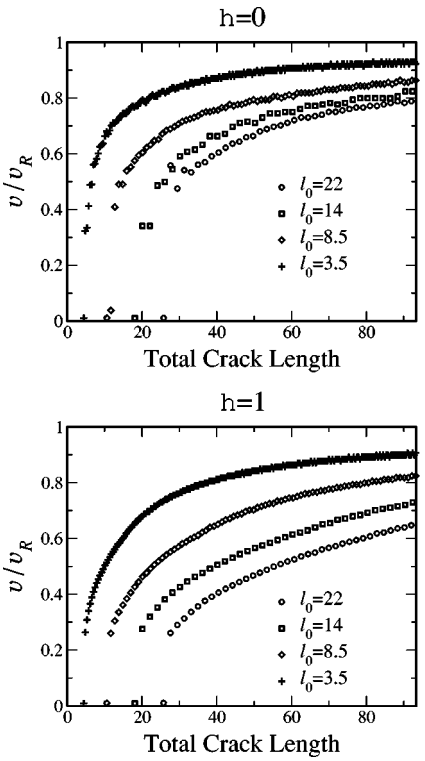

FIG. 1. Simulation results for velocity vs crack length for various initial seed cracks. The system widths W in lattice units are 239, 219, 415, and 1004 for the four runs for decreasing ℓ_0. For later convenience, all lengths in the graph are scaled by a factor 440/W. Velocities are normalized to the Rayleigh velocity.

$$\Gamma(v)=f(v)B_I^2. \tag{3}$$

Here f is a complicated but explicit function and Γ is the breaking energy. In fact, Γ can be determined via the dependence of steady-state cracks on the driving load and thereafter used for the accelerating crack case. For our purposes, this hypothesis leads to the existence of a universal relationship, independent of W, Δ, and ℓ_0 (but dependent on η), between the measured velocity $v(t)$ and the Eshelby function $B_0(\ell(t))$.

To test this strong prediction of LEFM, we use our lattice model simulations discussed above. We calculate the integral in Eq. (2) by replacing the continuum stress with its lattice analog [defining $\vec{u}\equiv(u,v)$],

$$\sigma_{eq}^{yy}(x)=-k_1 v_{eq}(x,0)-\frac{1}{2}k_2[u_{eq}(x-1,0)-u_{eq}(x+1,0)+v_{eq}(x-1,0)+v_{eq}(x+1,0)]. \tag{4}$$

The fields that enter are the equilibrium fields present before the crack tip begins to move. The integral is then replaced by a sum, taking care to resolve the singularities in the integrand. This yields a function $B_0(\ell;W,\ell_0)$ which can be plotted vs the velocity. The results of this exercise are pre-

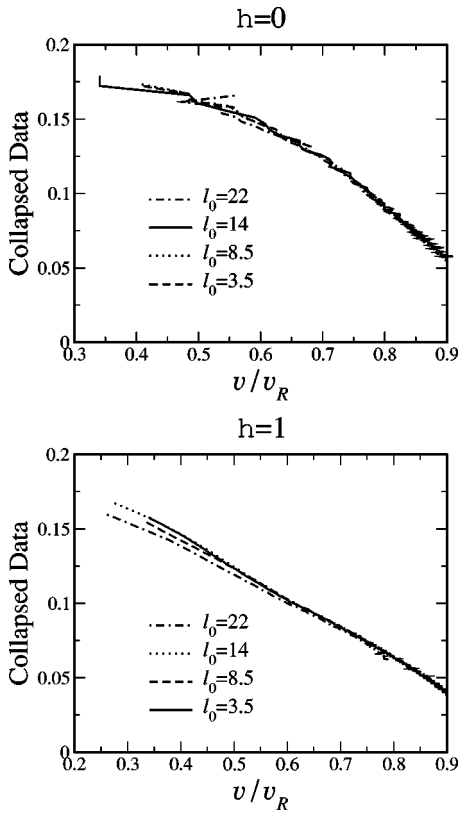

FIG. 2. Eshelby data collapse, $1/B_0$ vs v for the runs shown in Fig. 1 (both $\eta=0$ and $\eta=1$). Notice the slight systematic dependence on ℓ_0 for the $\eta=1$ data, presumably arising from the larger process zone in this case [9].

sented in Fig. 2, demonstrating almost perfect data collapse even for fairly small transverse sizes W. Thus, the LEFM assumption that the nonlinear structure of the process zone is only dependent on the velocity and the SIF is borne out in the lattice model.

Given that the lattice model does indeed serve as an instantiation of the LEFM picture developed by the fracture community for straight accelerating cracks, the question now is whether this carries over to the real world of experiment. For this reason, we designed our simulations to reproduce the large-scale features of the experiments on poly(methyl methacrylate) (PMMA) carried out by Sharon and Fineberg. Here too the loading is set to a point where a small perturbation (a touch of a razor blade) is enough to cause a previously created notch in the material to develop into a running crack. For PMMA, the elastic constants are $\lambda=2680$ MPa, $\mu=1150$ MPa, giving $\nu=0.35$. Going to the plane stress case leads to an effective Poisson ratio $\nu_{2d}\simeq 0.259$, which is very close to the value forced upon us by the restriction to nn and nnn central forces. (We did check that generalizing our model to include bond-bending springs and thereby obtaining the precise value of the Poisson ratio did not alter in any way the results which we will present below.) In addition, the aspect ratio was fixed to that of the experiment (a width of 440 mm and a length of 380 mm). The lengths of the initial seed cracks were also chosen to be in the same ratio to the width as in the experiment. This constraint fixed our choice

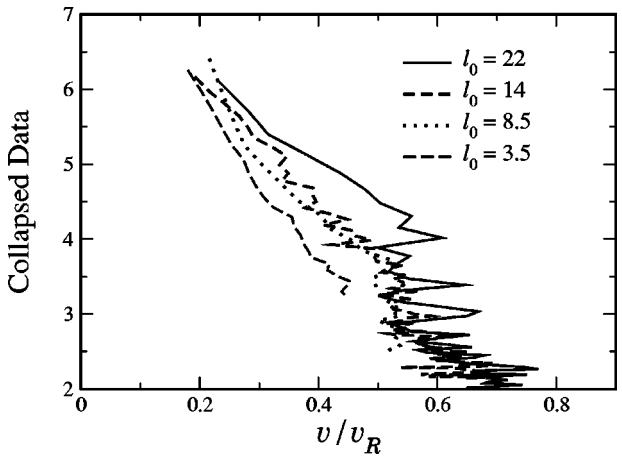

FIG. 3. Attempted data collapse for the Sharon-Fineberg PMMA experiment. Initial crack lengths are given in mm.

of the width in lattice units, as we wanted the initial crack to be large (of order 10) on the lattice scale.

Now, it is a simple matter to use the Eshelby B_0 function calculated as described above to process the actual crack velocity and length data from the Sharon-Fineberg experiments. Figure 3 shows our attempt to verify the continuum hypothesis for the PMMA data; note that each graph is labeled by the initial crack length which via the experimental protocol is in one-to-one correspondence with the total applied energy. In our opinion, the data are quite convincingly inconsistent with the universality of the v-B_0 relationship. Note that this is the opposite conclusion from that reached by the experimentalists themselves, as discussed in Ref. [15]. There, the exact Eshelby function was replaced by an approximate form based on the *static* stress-intensity factor for a crack of length $\ell(t)$. Note that these functions, while close for $\ell\approx\ell_0$, become quite different for larger length cracks. The static stress-intensity factor saturates for $\ell\gg W/2$, whereas the exact Eshelby function increases linearly in this regime. We have checked that the use of this approximation happens to push the data into closer agreement with the theory and could lead to a mistaken impression of data collapse. This does not happen with the exact expression and conversely the approximate form does not lead to a very good data collapse for the simulation study discussed above. The same situation pertains to the Sharon-Fineberg measurements on fracture in glass (results not shown). There the data collapse is, if anything, even worse.

It should be noted that the agreement between the lattice theory and LEFM is very dependent on our absolute suppression of off-axis cracking. If all bonds are allowed to break, then above a critical velocity the crack no longer propagates in a straight line, and the assumptions implicit in the Eshelby calculation of the instantaneous SIF break down completely. At that point, there is no longer any way to reliably measure the SIF and thus test LEFM. Needless to say, LEFM in its simplest formulation does not predict the direction of branching, and thus cannot address to the post-instability dynamics of the crack. The effects of the instability in the experiment (which appear to exhibit different dynamics than in the lattice model in this regime) are clearly visible at later

times (larger velocities), but cannot explain the apparent failure of the data collapse at smaller velocities.

The question remains how to explain this failure of LEFM in the Sharon-Fineberg experiment. Putting aside for the moment the obvious possibility of systematic errors in the determination of the velocity or position of the crack, or alternatively of the existence of an incredibly large process zone, the only simple explanation that suggests itself is that reflection from the free edge of the sample is sufficient to render inaccurate the Eshelby calculation of the stress-intensity factor. In support of this possibility, we determined that all the experimental data (except for the first point or two) are taken from times after a boundary-reflected wave (emitted initially as the tip begins accelerating) would catch up to the tip. Also, we checked that the lattice model does indeed deviate from universality if we include data from after this time [17]. Thus, our tentative conclusion is that the current experiments offer no direct proof for the LEFM approach, unlike what was claimed in Ref. [15]. We are currently pursuing a method that accounts for the side-reflected wave, and will allow us to test whether this is the cause of the failure of the data collapse.

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